POLYNOMIAL LEMNISCATES AND THEIR FINGERPRINTS: FROM GEOMETRY TO TOPOLOGY

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Abstract. We study shapes given by polynomial lemniscates, and their fingerprints. We focus on the inflection points of fingerprints, their number and geometric meaning. Furthermore, we study dynamics of zeros of lemniscate-generic polynomials and their 'explosions' that occur by planting additional zeros into a defining polynomial at a certain moment, and then studying the resulting deformation. We call this dynamics *polynomial fireworks* and show that it can be realized by a construction of a non-unitary operad.

1. Introduction

A polynomial lemniscate is a plane algebraic curve of degree $2n$, defined as a level curve of the modulus of a polynomial $p(z) = \prod_{k=1}^{n}(z - z_k)$ of degree n with its roots z_k called the nodes of the lemniscate. Lemniscates have been objects of interest since 1680 when they were first studied by French-Italian astronomer Giovanni Domenico Cassini see, e.g., [14], and later christened as "ovals of Cassini". 14 years later Jacob Bernoulli, unaware of Cassini's work, described a curve forming 'a figure 8 on its side', which is defined as a solution of the equation

$$(x^2 + y^2)^2 = 2c^2(x^2 - y^2), \quad \text{or } |z - c|^2|z + c|^2 = c^4, \tag{1}$$

where $z = x+iy$. (Curiously, his brother Johann independently discovered lemniscate in 1694 in a different context.) Observe that the level curves $|p(z)| = R$ can form a closed Jordan curve for sufficiently large R, and split into n disconnected components when $R \to 0^+$. Great interest in lemniscates in 18th century was inspired by their links to elliptic integrals. The dynamics of a lemniscate as R grows is illustrated in Figure 1.

The recent revival of lemniscates owes to the newly emerging field of vision and pattern recognition. The key idea is to consider the space of 2D 'shapes',

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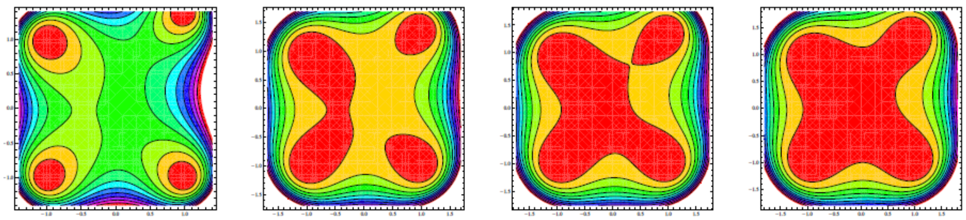

FIGURE 1. Evolution of a lemniscate of a polynomial of degree 4

domains in the complex plane $\mathbb{C}$ bounded by smooth C^∞ Jordan curves Γ dividing $\mathbb{C}$ into two simply connected domains one of which Ω_- contains infinity and the other one, Ω_+, is bounded. The study of an enormous space of 2D shapes was one of the central problems in a program outlined by Mumford at the ICM 2002 in Beijing [35]. We will also call the boundary curves Γ shapes for convenience.

Here we focus on 'fingerprints' of shapes obtained by means of conformal welding. Let Γ be a curve defining a shape Ω_+, and let ϕ_+ be a conformal mapping of the unit disk $\mathbb{D} = \mathbb{D}_+$ onto the domain Ω_+ bounded by Γ. The matching function ϕ_- maps the exterior $\mathbb{D}_-$ of $\mathbb{D}$ onto Ω_-. One can either normalize the interior maps by shifting and scaling Γ so that $0 \in \Omega_+$ and the conformal radius of Ω_+ with respect to the origin is 1, or the exterior one by the 'hydrodynamical' normalization

$$\phi_-(z) = z + \sum_{k=1}^{\infty} \frac{c_k}{z^k} \in \mathcal{G},$$

where by $\mathcal{G}$ we denote the class of all analytic functions in $\mathbb{D}_-$ normalized as above and C^∞ smooth up to the boundary. The conformal welding produces the function $k\colon [0, 2\pi] \to [0, 2\pi]$ defined by $e^{ik(\theta)} = (\phi_-)^{-1} \circ \phi_+(e^{i\theta})$, which is monotone, smooth and '2π-periodic' in the sense that $k(\theta + 2\pi) = k(\theta) + 2\pi$. In the first case, the function $e^{ik(\theta)}$ is a representative of an element of the smooth Teichmüller space $\text{Diff}\, S^1/\text{Rot}$, i.e., the connected component of the identity of the quotient Lie-Fréchet group $\text{Diff}\, S^1$ of orientation preserving diffeomorphisms of the unit circle S^1 over the subgroup of rotations Rot . In the second case, $e^{ik(\theta)}$ is an element of the smooth Teichmüller space $\text{Diff}\, S^1/\text{Möb}$, where Möb is the group $\text{PSL}(2, \mathbb{C})$ restricted to S^1. The fibration $\pi\colon \text{Diff}\, S^1/\text{Rot} \to \text{Diff}\, S^1/\text{Möb}$ has the typical fiber $\text{Möb}/S^1 \simeq \mathbb{D}_-$. The homogeneous spaces $\text{Diff}\, S^1/\text{Möb}$ and $\text{Diff}\, S^1/\text{Rot}$ carry the structure of infinite-dimensional, homogeneous, complex analytic Kählerian manifolds which appeared in the classification of orbits in the coadjoint representation of the Virasoro-Bott group see, e.g., [27, 28]. There is a biholomorphic isomorphism between $\mathcal{G}$ and $\text{Diff}\, S^1/\text{Möb}$, see [27].

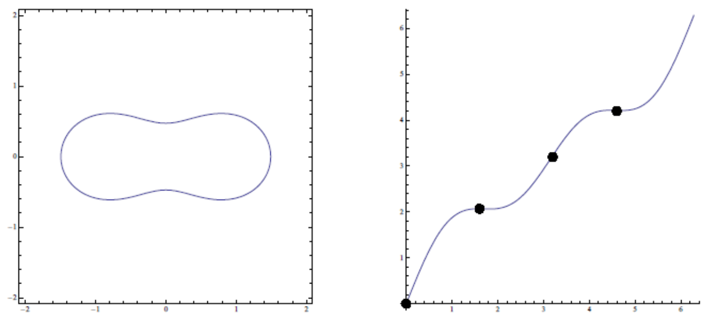

FIGURE 2. Cassini oval (Bernoulli's lemniscate of degree 2) and its fingerprint (the marked points are the points of inflection).

The construction of fingerprints is straightforward whereas the reconstruction of shapes from their fingerprints is a non-trivial task. Let us mention two more recent reconstructing algorithms: one provided by Mumford and Sharon [42], the second is the 'zipper' algorithm by Marshall [30, 31].

By Hilbert's theorem, polynomial lemniscates approximate any smooth shape with respect to the Hausdorff distance in the plane, see e.g., [21] and [46, Ch. 4]. An advantage of this approach is that the fingerprint of a polynomial lemniscate is given by the n^{th} root of a Blaschke product of degree n, which was proved in [17], see also [48] for a simplified proof and extensions to rational lemniscates. The reciprocal statement also holds: a fingerprint given by an $n-$th root of a Blaschke product of degree n represents the shape given by a polynomial lemniscate of the same degree [17, 48, 41]. Also, cf. [41] for some generalizations.

The proof of Hilbert's theorem was based on approximation of the Riemann integral by the Riemann sums and the intermediate points of the partitions were chosen as the nodes of the approximating lemniscates. This approach is algorithmically poor. Indeed, the more precise approximation we need, the higher degree of the polynomial we must choose. So, even starting with a shape already defined by a lemniscate of a low degree, the approximating lemniscate will be of much higher degree. This problem was addressed by Rakcheeva in [38, 39]. She proposed a focal algorithm which starts with a just a polynomial of degree 1, and then, 'budding' the zeros of the polynomial iteratively according to the shape, a better approximation is achieved. Based on this idea we also start with a lemniscate for the polynomial of degree 1, just a circle. Then we place a power $z^{n_k^j}$ at a simple zero z_k during the j-th step of the iteration, and then, perform a deformation, i.e., we move the new zeros without significant change in the structure of lemniscates. From the algebraic viewpoint, as a result, we arrive at a braid operad which we call the *polynomial* *fireworks operad*. This way, smooth shapes encode the polynomial fireworks operad.

The structure of the paper is as follows. First, we observe some simple properties of real analytic shapes and shapes with corners. Then, we study the role of inflection points of a fingerprint and their relation to the corresponding shape. Finally, after presenting some necessary definitions and some background we outline the construction of the polynomial fireworks operad.

Note. Professor Vasil'ev (Sasha to his many friends and colleagues) conceived the idea of this work and enthusiastically worked towards its completion. The untimely death didn't let him to see the final version. We shall all miss him, his friendship, insights and his kindness.

2. On the geometry of fingerprints

In this section we describe some simple relations between shapes and their fingerprints, which we couldn't find in the literature.

2.1. **Real-analytic fingerprints.** As is shown in [27, 28], any smooth increasing function $k\colon [0,2\pi]\to[0,2\pi]$, satisfying $k(\theta+2\pi)=k(\theta)+2\pi$, defines a smooth shape Γ. However, if Γ is real analytic, it restricts the fingerprints severely. Indeed, if an analytic shape Γ contains a circular arc, then Γ is a circle. Furthermore, Γ is bounded, and therefore, a real analytic Γ can not contain a line segment. At the same time, periodicity of $k(\theta)$ implies that the fingerprint in the square $[0,2\pi]\times[0,2\pi]$ can not contain a segment of a straight line unless $k(\theta)=\theta+const$.

Theorem 1. *Let $p(z)$ be a polynomial of degree at least two. If the fingerprint $k(\theta)$ of a shape Γ is given by the relation $e^{ik(\theta)}=p(e^{i\theta})$ in some closed interval $\sigma\subseteq[0,2\pi]$, where $p(z)$ is a polynomial, then Γ is not an analytic curve.*

Proof. Indeed, if Γ were analytic, then its fingerprint $k\in \mathrm{Diff}\, S^1/\text{Möb}$ would be given by the relation $e^{ik(\theta)}=(\phi_-)^{-1}\circ\phi_+(e^{i\theta})$, where

$$\phi_+(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n, \quad \text{and} \quad \phi_-(\zeta)=\zeta+\sum_{n=1}^{\infty}\frac{b_n}{\zeta^n}.$$

Both functions have continuous extension to S^1. Moreover, $\phi_-\circ p(e^{i\theta})$ and $\phi_+(e^{i\theta})$ have analytic extensions to a neighbourhood of the arc $\{z\colon z=e^{i\theta},\,\theta\in\sigma\}$. Thus, by Morera's theorem, $\phi_-\circ p(e^{i\theta})$ and $\phi_+(e^{i\theta})$ are analytic extensions of each other, hence, equating the coefficients, we conclude that all coefficients a_n and b_n must vanish except for the case $k(\theta)=\theta$. A simple adjustment of the normalization of $\phi_+(e^{i\theta})$ yields the proof in the case $k\in \mathrm{Diff}\, S^1/\mathrm{Rot}$. $\square$

2.2. **Shapes with corners.** Let us now remark on the case when the boundary curve Γ has a corner of opening $\pi\alpha$ $(0 \le \alpha \le 2)$ at a point $z_\alpha \in \Gamma$. Then the interior mapping ϕ_+ satisfies the condition

$$\arg[\phi_+(e^{it}) - \phi_+(e^{i\theta_\alpha})] \to \begin{cases} \beta & \text{as } \ t \to \theta_\alpha + 0, \\ \beta + \pi\alpha & \text{as } \ t \to \theta_\alpha - 0, \end{cases}$$

where $z_\alpha = \phi_+(\theta_\alpha)$. If $\alpha = 1$, then Γ has a tangent at z_α, if $\alpha = 0$ or $\alpha = 2$, then Γ has an outward-pointing or an inward-pointing cusp respectively. If Γ is smooth except at the point z_α, $\alpha \in (0, 1)$, then the derivative $(\phi_+)'(\zeta)$ has a continuous extension to $S^1 \setminus \{e^{i\theta_\alpha}\}$, and the functions $(\zeta - e^{i\theta_\alpha})^{1-\alpha}(\phi_+)'(\zeta)$ and $(\zeta - e^{i\theta_\alpha})^{-\alpha}(\phi_+(\zeta) - \phi_+(e^{i\theta_\alpha}))$ are continuous in some neighbourhood of $e^{i\theta_\alpha}$ in $\hat{\mathbb{D}}$, see e.g., [37, Theorem 3.9]. Similar conclusions hold for the exterior mapping ϕ_-. The fingerprint $k(\theta)$ does not longer represent an element of Diff S^1/Rot or Diff $S^1/\text{Möb}$ but it belongs to Hom S^1/Rot or Hom $S^1/\text{Möb}$, where Hom denotes a group of quasisymmetric homeomorphisms of S^1. Since $\phi_-(e^{ik(\theta)}) = \phi_+(e^{i\theta})$, we have

$$\begin{aligned} \phi_-(\zeta) &= z_\alpha + b_1(\zeta - e^{ik(\theta_\alpha)})^{2-\alpha} + o(|\zeta - e^{ik(\theta_\alpha)}|^{2-\alpha}) \\ \phi_+(\zeta) &= z_\alpha + d_1(\zeta - e^{i\theta_\alpha})^{\alpha} + o(|\zeta - e^{i\theta_\alpha}|^{\alpha}) \end{aligned}$$

in the corresponding neighbourhoods of the points $e^{ik(\theta_\alpha)}$ and $e^{i\theta_\alpha}$ in $\hat{\mathbb{D}}^-$ and $\hat{\mathbb{D}}$ respectively. After performing the conformal welding it is clear that the original fingerprint $k(\theta)$ has the singular point $\theta_\alpha \in [0, 2\pi)$, i.e., the graph has a singularity ($k'(\theta_\alpha) = \infty$ when $\alpha = 2$) of order $2(\alpha - 1)/(2 - \alpha)$, $k'(\theta) \sim |\theta - \theta_\alpha|^{\frac{2(\alpha-1)}{2-\alpha}}$.

2.3. **Dynamics of proper lemniscates when they are approaching critical points.** Let $\Gamma(R)$ be a polynomial lemniscate of degree n, i.e.,

$$\Gamma(R) = \{z \in \mathbb{C} \colon |p(z)| = R\}, \quad p(z) = \prod_{k=1}^{n}(z - z_k), \quad R > 0.$$

Without loss of generality let us, whenever possible, assume $R = 1$. Let us denote the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The lemniscate $\Gamma(1)$ is called *proper* if the region $\Omega^+ = \{z \in \hat{\mathbb{C}} : |p(z)| < 1\}$, is connected. Let us denote by Ω^- the unbounded component of $\hat{\mathbb{C}} \setminus \Gamma(1)$, i.e. $\Omega^- = \{z \in \hat{\mathbb{C}} : |p(z)| > 1\}$. If $\Gamma(1)$ is proper, the Riemann map $\phi_- \colon \mathbb{D}_- \to \Omega_-$ has a simple inverse $(\phi_-)^{-1}(z) = \sqrt[n]{p(z)}$. Let $\{y_j\}_{j=1}^{n-1}$ denote the *critical points* of p, i.e. the zeros of the derivative $p'(z)$. Then $\Gamma(1)$ is proper if and only if all the critical points $\{y_j\}_{j=1}^{n-1}$ of p lie in Ω^+, or, equivalently, all the critical values $p(y_j)$ lie in $\mathbb{D}_+$, cf. [17].

Let $B(\zeta)$ stand for the Blaschke product

$$B(\zeta) = e^{i\alpha} \prod_{k=1}^{n} \frac{\zeta - a_k}{1 - \bar{a}_k \zeta},$$

for some real α and $|a_k| < 1$. Then Theorem 2.2 from [17] states that the fingerprint $k\colon [0, 2\pi] \to [0, 2\pi]$ of the proper lemniscate $\Gamma(1)$ is given by

$$e^{ik(\theta)} = \sqrt[n]{B(e^{i\theta})}, \tag{2}$$

where the branch of $\sqrt[n]{\cdot}$ is appropriately chosen, e.g., by fixing the branch $\sqrt[n]{1} = 1$, and the zeros a_k of $B(\zeta)$ are the pre-images of the nodes z_k under ϕ_+, repeated according to the multiplicity.

Let us discuss the dynamics of proper lemniscates $\Gamma(R)$, $R > |p(y_{n-1})|$, as $R \searrow |p(y_{n-1})|$, where we first assume that $|p(y_k)|$, $k = 1, \dots, n-1$ are the modules of the critical values of p at the points y_k, $y_{n-1} \neq y_k$ and $|p(y_{n-1})| > |p(y_k)|$, $k = 1, \dots, n-2$. When the proper lemniscate approaches the first critical point, and the domain Ω_+ splits up into exactly two domains with the multiple boundary point of valence 4. Let $k(\theta)$ represent an element of $\mathrm{Diff}\, S^1/\text{Möb}$. If $R \searrow |p(y_{n-1})| + 0$, then the exterior mapping ϕ_- still exists at the limit point. In order to give any reasonable sense to what happens with the interior conformal map let us use the Carathéodory theorem, see e.g., [37, Theorem 1.8] fixing a point in one of the parts Ω_1 or Ω_2 of Ω_+ bounded by $\Gamma(|p(y_{n-1})|)$. For example, we can specify one of the nodes of the lemniscate $z_1 \in \Omega_1$ as the image of the origin by ϕ_+, $\phi_+(0) = z_1$, for all $R > |p(y_{n-1})|$. Then, the preimages $a_1, \dots, a_m$ of those nodes $z_1, \dots, z_m$ that remain in Ω_1 will remain in $\mathbb{D}$, while all other preimages $a_{m+1}, \dots, a_n$ of the nodes $z_{m+1}, \dots, z_n \in \Omega_2$ will tend to S^1 as $R \searrow |p(y_{n-1})|$. At the same time, the preimage of Ω_2 will collapse and the Carathéodory convergence theorem guarantees that the limiting map ϕ_+ will be well-defined in $\mathbb{D}$ (the kernel), $\phi_+\colon \mathbb{D} \to \Omega_1$, as $R \searrow |p(y_{n-1})|$. The inverse $(\phi_-)^{-1}$ of the exterior map can be continuously (non-bijectively) extended to $\partial\Omega_1$, where the bifurcation point is understood to be two different points over the same support. Then the fingerprint $k(\theta)$ can be made sense of only between two points corresponding to the images of the bifurcation point. That is, as in the previous section, the graph of $k(\theta)$ will have the vertical tangent at the points $\theta^1_{1/2}$ and $\theta^2_{1/2}$, $\theta \in [\theta^1_{1/2}, \theta^2_{1/2}]$, and the order of the singularity is $(-3/2)$, as the lemniscate has a corner of opening $\pi/2$ at the singular point.

This argument can be generalized to the case $|p(y_{n-1})| \geq |p(y_k)|$, $k = 1, \dots, n-2$ and y_{n-1} is allowed to coincide with other critical points. In this case, the domain Ω_1 containing a fixed node can have several angular points of different angles and the fingerprint will have several singularities of different orders.

Let us mention in passing that similar arguments apply to Diff S^1/Rot as long as we specify which domain bounded by the critical lemniscate we consider as the Carathéodory kernel. This can be achieved, e.g., by considering equivalent polynomials. (Recall that two polynomials p_1 and p_2 are said to be from the same conjugacy class $[p]$ if there is an affine map A such that $p_2 = A^{-1} \circ p_1 \circ A$. In this case the geometry of the lemniscates of p_1 and p_2 is the same up to scaling, translation and rotation, i.e., as 'shapes' those lemniscates are indistinguishable.)

3. Nodes of lemniscates and inflection points of fingerprints

The first feature one observes looking at a fingerprint of a smooth shape is that it possesses a number of inflection points. It turns out that lemniscates' fingerprints must have at least two inflection points. More precisely, the following is true, cf. Figure 2.

Theorem 2. *The fingerprint $k(\theta)$ given by (2) has an even number of inflection points, at least two and at most $4n-2$.*

Proof. If we write $a_k = |a_k|e^{i\theta_k}$, then

$$k'(\theta) = -\frac{i}{n}\frac{\partial}{\partial\theta}\log B(e^{i\theta}) = \frac{1}{n}\sum_{k=1}^{n}\frac{1-|a_k|^2}{1+|a_k|^2-2|a_k|\cos(\theta_k-\theta)}$$

or, in terms of the Poisson kernel,

$$k'(\theta) = \frac{1}{n}\mathrm{Re}\,\sum_{k=1}^{n}\frac{\zeta+a_k}{\zeta-a_k}, \quad \zeta = e^{i\theta}.$$

Respectively,

$$k''(\theta) = \frac{1}{n}\mathrm{Re}\,\sum_{k=1}^{n}\frac{-2a_k i\zeta}{(\zeta-a_k)^2} = -\frac{1}{2n}\sum_{k=1}^{n}\left(\frac{2a_k i\zeta}{(\zeta-a_k)^2}-\frac{2\bar{a}_k i\zeta}{(1-\bar{a}_k\zeta)^2}\right).$$

First, observe that the rational function $\sum_{k=1}^{n}\frac{\zeta+a_k}{\zeta-a_k}$ maps the unit circle onto a smooth closed curve with possible self-intersections. Hence, its real part $k'(\theta)$ attains at least one maximum and one minimum. Therefore, the function $k''(\theta)$ has at least two zeros in $[0,2\pi)$ at which $k''(\theta)$ changes the sign. An elementary calculus theorem states that for the graph of a differentiable function of one variable $f(x)$, the number of points c where f has a local extremum in the interval $[a,b]$ is even if $f'(a)$ and $f'(b)$ have the same sign and this number is odd if the signs are different, assuming that there is a finite number of critical points in the interval, and that the sign of the derivative changes as we go from left to right passing through a zero of the first derivative. Consider finite alternating sequences of, let say, (+) and (-). If one has an equal number of alternating (+)'s and (-)'s, then the sign changes odd number of times, and if the number of alternating (+)'s and (-)'s differs by 1, then the sign changes even number of times.

Hence, the periodic function k' has an even number of critical points corresponding to the inflection points of k.

At the same time, the rational function

$$Z(\zeta) = \zeta \sum_{k=1}^{n} \left(\frac{2ia_k}{(\zeta - a_k)^2} - \frac{2i\bar{a}_k}{(1 - \bar{a}_k\zeta)^2} \right)$$

has degree $4n$, has simple zeros at the origin and ∞, and satisfies the relation $\bar{Z}(\zeta) = Z(1/\bar{\zeta})$. Therefore, if $b_1, \dots, b_m$ are zeros of the function $Z(\zeta)$ in $\mathbb{D} \setminus \{0\}$, then $1/\bar{b}_1, \dots, 1/\bar{b}_m$ are also its zeros in $\mathbb{D}_- \setminus \{\infty\}$, so $0 \le m \le 2n-1$. The rest of $4n - 2 - 2m$ zeros of Z lie on the unit circle and are precisely the zeros of the function k''. □

However, the maximal number $4n - 2$ of the inflection points need not be achieved. The following theorem provides a more detailed explanation of this phenomenon.

Theorem 3. *If all nodes of the n-Blaschke product B lie on the same radius of $\mathbb{D}$, then the number of the inflection points of the fingerprint k, $e^{ik(\theta)} = \sqrt[n]{B(e^{i\theta})}$ is at most $4n-4$. In the particular case when $n = 2$, the number of the inflection points is at most 4 (not 6!) for arbitrary position of the nodes of B.*

Proof. Set

$$\Psi(\zeta) = \prod_{k=1}^{n} (1 - \bar{a}_k\zeta)^2 (\zeta - a_k)^2,$$

and define the polynomial P of the form

$$P(\zeta) = \Psi(\zeta) \sum_{j=1}^{n} \left(\frac{a_j}{(\zeta - a_j)^2} - \frac{\bar{a}_j}{(1 - \bar{a}_j\zeta)^2} \right).$$

Then

$$\frac{P'(0)}{P(0)} = \frac{\Psi'(0)}{\Psi(0)} + 2\frac{\sum_{k=1}^{n}\left(\frac{1}{a_k^2} - \bar{a}_k^2\right)}{\sum_{k=1}^{n}\left(\frac{1}{a_k} - \bar{a}_k\right)} =$$

$$= -2\sum_{k=1}^{n}\left(\frac{1}{a_k} + \bar{a}_k\right) + 2\frac{\sum_{k=1}^{n}\left(\frac{1}{a_k^2} - \bar{a}_k^2\right)}{\sum_{k=1}^{n}\left(\frac{1}{a_k} - \bar{a}_k\right)} =$$

$$= 2\frac{\sum_{k=1}^n \left(\frac{1}{a_k^2} - \bar{a}_k^2\right) - \sum_{k=1}^n \left(\frac{1}{a_k} + \bar{a}_k\right) \sum_{k=1}^n \left(\frac{1}{a_k} - \bar{a}_k\right)}{\sum_{k=1}^n \left(\frac{1}{a_k} - \bar{a}_k\right)} =$$

$$= -2\ \frac{\sum_{1\le k\neq j\le n} \left(\frac{1}{a_k} + \bar{a}_k\right)\left(\frac{1}{a_j} - \bar{a}_j\right)}{\sum_{k=1}^n \left(\frac{1}{a_k} - \bar{a}_k\right)} =$$

$$= -4(n-1)\ \frac{\displaystyle\sum_{1\le k<j\le n} \frac{1-|a_k a_j|^2}{a_k a_j}}{\displaystyle\sum_{1\le k<j\le n} \frac{a_j(1-|a_k|^2) + a_k(1-|a_j|^2)}{a_k a_j}}.$$

If all a_k lie on the same radius, then we have

$$\left|\frac{P'(0)}{P(0)}\right| > 4(n-1); \tag{3}$$

and ± 1 are among the roots. Therefore, we have $|\frac{P'(0)}{P(0)}| = 2|\sum_{k=1}^{2n-2}\cos\theta_k| \le 4n-4$, which contradicts (3) and finishes the proof of the first statement of the theorem.

In the case $n = 2$,

$$\frac{P'(0)}{P(0)} = \frac{4(1-|a_1|^2|a_2|^2)}{a_1(1-|a_2|^2) + a_2(1-|a_1|^2)},$$

so we have

$$\left|\frac{P'(0)}{P(0)}\right| \ge \frac{4(1-|a_1|^2|a_2|^2)}{|a_1|(1-|a_2|^2) + |a_2|(1-|a_1|^2)} = \frac{4(1+|a_1||a_2|)}{|a_1|+|a_2|} > 4. \tag{4}$$

The polynomial $P(\zeta)$ is self-inversive because $\zeta^6\bar{P}(1/\bar{\zeta}) = -P(\zeta)$. Therefore,

- If b is a root, then $1/\bar{b}$ is also a root;
- If $e^{i\theta}$ is a root, then $e^{-i\theta}$ is also a root.

If $b_1, \dots, b_6$ are the roots of $P(\zeta)$, then, by Vieta's theorem, and recalling that $P(\zeta)$ is self-inversive,

$$\sum_{k=1}^{6} b_k = \overline{\frac{P'(0)}{P(0)}}.$$

Let us assume that there are exactly six zeros $e^{\pm i\theta_k}$, $k = 1, 2, 3$, in S^1. Then

$$\overline{\frac{P'(0)}{P(0)}} = 2(\cos\theta_1 + \cos\theta_2 + \cos\theta_3),$$

and so $P'(0)/P(0)$ is real. There are two possibilities:

(1) $|a_1| = |a_2|$ and $a := a_1 = \bar{a}_2$;
(2) $|a_1| \neq |a_2|$ and a_1, a_2 are real.

In the first case,

$$\begin{aligned} P(\zeta) &= a(1-\bar{a}\zeta)^2(1-a\zeta)^2(\zeta-\bar{a})^2 \\ &+ \bar{a}(1-\bar{a}\zeta)^2(1-a\zeta)^2(\zeta-a)^2 \\ &- \bar{a}(1-a\zeta)^2(\zeta-a)^2(\zeta-\bar{a})^2 \\ &- a(1-\bar{a}\zeta)^2(\zeta-a)^2(\zeta-\bar{a})^2, \end{aligned}$$

and this polynomial has ± 1 among the roots. Therefore, $\overline{\frac{P'(0)}{P(0)}} = 2\cos\theta_3$, which contradicts (4).

In the second case,

$$\begin{aligned} P(\zeta) &= a_1(1-a_1\zeta)^2(1-a_2\zeta)^2(\zeta-a_2)^2 \\ &+ a_2(1-a_1\zeta)^2(1-a_2\zeta)^2(\zeta-a_1)^2 \\ &- a_1(1-a_2\zeta)^2(\zeta-a_1)^2(\zeta-a_2)^2 \\ &- a_2(1-a_1\zeta)^2(\zeta-a_1)^2(\zeta-a_2)^2, \end{aligned}$$

and this polynomial has again ± 1 among the roots , which contradicts (4) for the same reason.

Summarizing, the polynomial $P(\zeta)$ has at least one root in $\mathbb{D}_+$ and hence, another in $\mathbb{D}_-$, and the maximal number of the inflection points in the fingerprint of a Bernoulli lemniscate is 4. $\square$

Remark 1. *The upper bound 4 of zeros of the function $Z(\zeta)$ for $n = 2$ is achieved. For example, if $a_1 = -1/2$, and $a_2 = 1/2$, then the function has a double zero at the origin and at infinity, and four zeros $1, i, -1, -i$ on the circle S^1.*

The following statement clarifies the geometric meaning of the fingerprints' inflection points in general setting.

Theorem 4. *The inflection points of the fingerprint $k(\theta)$ divide the unit circle S^1 into m arcs $\gamma_j = \{e^{i\theta} \colon \theta \in [\theta_j, \theta_{j+1})\}$, $\theta_{m+1} = \theta_1 + 2\pi$, where $j = 1, \dots, m$, so that the ratio of the rates of change of the harmonic measures of the arc $\alpha \subset \Gamma$, $\alpha = \{\phi_+(s) \colon s \in [\theta_1, \theta)\}$ with respect to $(\Omega^+, 0)$ and (Ω^-, ∞) respectively, alternates its monotonicity.*

Proof. The fingerprint $k(\theta)$ of a curve α is defined by

$$e^{ik(\theta)} = \phi_+^{-1} \circ \phi_-(e^{i\theta}).$$

We rewrite the last expression as

$$\phi_+(e^{ik(\theta)}) = \phi_-(e^{i\theta}),$$

and differentiate it

$$k'(\theta)e^{ik(\theta)}\phi'_+(e^{ik(\theta)}) = \phi'_-(e^{i\theta})e^{i\theta}.$$

Without loss of generality we can assume that $\theta_1 = 0$, and

$$\phi_+(1) = \phi_-(1).$$

We consider an arc α on $\partial\Omega_+$ starting at the point $\phi_+(1)$.

Let γ_+ and γ_- denote the images of α by ϕ_+^{-1} and ϕ_-^{-1} correspondingly, which can be parametrized as follows: $\gamma_-(\theta) = e^{i\theta}$, $\gamma_+(\theta) = e^{ik(\theta)}$. Let us determine the harmonic measure

$$\omega_-(\alpha,\infty) = \int_\alpha \frac{\partial g(z,\infty)}{\partial n}|ds|$$

where $g(z,\infty)$ is Green's function, $g(z,\infty) = \operatorname{Re} G(z,\infty)$, and $G(z,\infty)$ is complex Green's function, that satisfies

$$G(z,\infty) = \log\phi_-^{-1}(z).$$

The normal derivative of $g(z,\infty)$ has form

$$\frac{\partial g(z,\infty)}{\partial n} = \operatorname{Re}\left(G'(z,\infty)\frac{e^{i\theta}\phi'_-}{|\phi'_-|}\right),$$

and thus the harmonic measure satisfies

$$\omega_-(\alpha,\infty) = \operatorname{Re}\int_\alpha G'(z,\infty)\frac{e^{i\theta}\phi'_-}{|\phi'_-|}|dz| =$$

$$\int_{\gamma_-} G'(\phi_-(\zeta),\infty)\zeta\phi'_-(\zeta)|d\zeta| = \int_{\gamma_-}|d\zeta| = \theta,$$

where $\zeta = e^{i\theta}$. We differentiate the resulting harmonic measure $\omega_-(\alpha,\infty)$

$$\frac{d}{d\theta}\omega_-(\alpha,\infty) = G'(\phi_-(e^{i\theta}),\infty)e^{i\theta}\phi'_-(e^{i\theta}).$$

Analogously, we obtain

$$\frac{d}{d\theta}\omega_+(\alpha,0) = G'(\phi_+(e^{ik(\theta)}),0)e^{ik(\theta)}\phi'_+(e^{ik(\theta)})k'(\theta).$$

Thus

$$k'(\theta)\frac{\partial}{\partial\theta}\omega_-(\alpha,\infty) = \frac{\partial}{\partial\theta}\omega_+(\alpha,0).$$

Therefore, $k'(\theta)$ shows the ratio of the rates of change of the respective harmonic measures.

$\square$

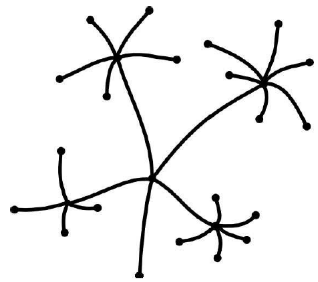

FIGURE 3. Polynomial fireworks

Remark 2. *We can now rephrase theorem 2 as follows. In the case of a lemniscate of degree n, the number m of points where the rates of change of interior and exterior harmonic measures change roles in dominating one another, is even and is at least 2 and at most $4n-2$.*

4. Polynomial fireworks

As it was observed in Introduction, the focal algorithm [38, 39] suggests a process of construction of lemniscates approximating smooth shapes by budding the new nodes, i.e., blowing up the old ones iteratively. So any smooth shape encodes a tree of evolution of lemniscate's nodes. In this section we study dynamics of zeros of lemniscate-generic polynomials and their explosion planting singularities at certain moment, and then, performing their deformation and evolution. We call this dynamics *polynomial fireworks* and it is realized by a construction of a non-unitary operad. The term is chosen because of the similarity to the real fireworks, cf. Figure 3.

4.1. **Trees.** Following [11], we call a polynomial $p(z)$ *lemniscate-generic* if the zeros $y_1, \dots, y_{n-1}$ of $p'(z)$ are distinct, $w_k = p(y_k) \neq 0$ for $k = 1, \dots, n-1$ and $|w_i| < |w_j|$ for $i < j$. Then, only finitely many level sets $\Gamma(R)$, $R > 0$ of $|p(z)|$ are not unions of 1-manifolds. Such a singular level set for a fixed R is called a *big lemniscate*. Each big lemniscate contains one singular connected component, i.e. a 'figure-eight', which is called a *small lemniscate*.

A lemniscate-generic polynomial $p(z)$ of degree n has exactly $n-1$ big and, correspondingly, small lemniscates. If p and p^* are lemniscate-generic polynomials of degree n, the unions Λ and Λ^* of all big lemniscates of p and p^* belong to the same *lemniscate configuration* $(\Lambda, \mathbb{C})$ if there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ mapping Λ onto Λ^*. Thus, we identify lemniscate configurations and the isotopy classes of lemniscate-generic polynomials.

Consider the set $\mathcal{P}_n$ of all polynomials of degree exactly n and the subset $\mathcal{L}_n$ of lemniscate-generic polynomials. Then $\mathcal{L}_n$ is open and $\mathcal{P}_n \setminus \mathcal{L}_n$ is a union of real hypersurfaces as it will be shown in Section 4.3. The lemniscate configuration does not change if p varies in a connected component E of $\mathcal{L}_n$.

If $[R_1, R_2]$ is an interval in $\mathbb{R}^+$ containing none of the points $|w_k| = |p(y_k)|$, where w_k are the critical values of p, then $\Gamma(R_1)$ is diffeomorphic to $\Gamma(R_2)$ by a gradient flow as it follows from Morse theory, see [34]. That is $|p|^2$ becomes a local Morse function whose Hessian matrix is non-degenerate at the critical points and whose gradient generates a flow between them.

Catanese and Paluszny [11] established a bijection between the connected components of $\mathcal{L}_n$ and the lemniscate configurations of polynomals of degree n. They also showed bijection between lemniscate configurations and simple central balanced trees of length $n-1$. Let us recall some terminology.

By a tree we understand a connected graph without cycles. A valence of a vertex is the number of edges adjacent to it. A vertex of valence 3 is called a node, a vertex of valence 1 is called an end. Any two vertices a and b of a tree can be connected by a path, i.e., a sequence of edges that connects the vertices. Any such path is in fact a sub-tree. The distance between a and b is the number of edges in a shortest path connecting a and b. A chain of edges is a tree that consists of subsequent edges $e_1, \dots, e_n$, such that e_k shares a vertex with e_{k+1}, $1 \le k \le n-1$.

A root radius of a vertex a is the maximal of distances from a to the leaves of the tree. A tree has vertex v as a center if v is a vertex with a minimal root radius. A tree is central if it has only one center. The length $|T|$ of a central tree T is the distance from the center of T to the ends of T. We call a tree binary if it only has vertices of valence no bigger than 3.

A tree T is called a simple central balanced tree of length $n-1$ if it has $n-1$ leaves, it is central, the root radius of the center v is $n-1$, the minimal of distances from v to the leaves is $n-1$; the valence of the center is 2, there is exactly one node at distance j from the center $(1 \le j \le n-2)$, and the tree does not have vertices of valence ≥ 3.

To each polynomial $p(z) \in \mathcal{L}_n$ we can assign a simple central balanced tree T of length $n-1$. The vertices of T of valence ≥ 2 represent connected components of big lemniscates. The leaves of T represent the zeros of p. Vertices of T at distance k from the center $(0 \le k \le n-2)$ represent connected components of a big lemniscate L_k: the nodes represent the figure-eight-like branchings and the vertices of valence 2 represent circumferences. T has $(n-1)(n-2)/2$ vertices corresponding to circumferences, $n-1$ vertices corresponding to the figure-eights, n vertices corresponding to zeros. The edges of T can be interpreted as the doubly connected domains situated between the connected components of critical level sets.

4.2. **Operad.** The notion of an operad first appeared and was coined in May's book [33] in 1972, and in the original work in algebraic topology by Boardman and Vogt [9], in the study of iterated loop spaces formalizing the idea of an abstract space of operations, see also more modern review in [29]. Thus, it is not surprising that operad appears in our case of shapes, i.e., 1-D loop space. Here is a precise definition.

Definition 1. *An operad is a sequence $\{\mathcal{O}(n)\}_{n=1}^{\infty}$ of sets (topological spaces, vector spaces, complexes, etc.), an identity element $e \in \mathcal{O}(1)$, and composition map $\circ$ defined for all positive integers $n; k_1, k_2, \dots, k_n$*

$$\begin{aligned} \circ\colon \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) &\to \mathcal{O}(k_1 + \cdots + k_n) \\ \theta, \theta_1, \dots, \theta_n &\to \theta \circ (\theta_1, \dots, \theta_n) := (\theta; \theta_1, \dots, \theta_n); \end{aligned}$$

satisfying the following axioms:

- *Associativity:*

$$\begin{aligned} &\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})) \\ = \;&(\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{n,1}, \dots, \theta_{n,k_n}). \end{aligned}$$

- *Identity: $\theta \circ (1, \dots, 1) = \theta = 1 \circ \theta$.*

Important examples of operads are the endomorphism operad, Lie operad, tree operad, 'little something' operad, etc.

A *non-unitary* operad is an operad without the identity axiom.

4.3. **Construction of polynomial fireworks.** The idea of the construction is as follows. We work with the space $\mathcal{M}_n$ of complex conjugacy classes $[p]$ of polynomials p of degree n where affine maps appear as a precomposition from the right and multiplication by a complex constant acts as a postcomposition from the left. Since any $p \in [p]$ belongs to the same connected component of $\mathcal{L}_n$, we will write simply p as a representative of $[p]$. The operation of composition of lemniscates (which will be used in the operad construction) consists of planting a zero of higher order in the place of the original zero and deforming it into simple zeros at the first moment.

Now the question is what happens analytically?

Take a polynomial $p \in [p] \in \mathcal{M}_n \subset \mathcal{L}_n$, and look at one of its zeros z_k. Let us consider a polynomial $q \in [q] \in \mathcal{M}_m \subset \mathcal{L}_m$. We want to define the operation $[p] \circ_k [q]$. Take another polynomial $\tilde{q} \in [q]$ such that all big lemniscates of $\tilde{q}$ are found inside the disk $U_r(z_k) = \{z \in \mathbb{C}\colon |z - z_k| < r\}$ for a sufficient small r such that U_r is inside the circular domain of the lemniscate configuration centered at z_k. Construct $\tilde{p} = (z - z_k)^{-1} p(z) \tilde{q}(z)$. Components of $\mathcal{L}_n$ are invariant under pre-composition with affine maps so we can assume without loss of generality that the polynomial p_n has one zero at 0 instead of z_k. If the polynomial $\tilde{p}$ is lemniscate generic, then the class $[\tilde{p}] \in \mathcal{M}_{n+m-1}$ will be the result of the superposition $[\tilde{p}] = [p] \circ_k [q]$. It is, of course, not true in general however it is always possible to find a path from the boundary point of $\mathcal{L}_{m+n-1}$ containing the non-lemniscate generic polynomial $z^{m-1}p(z)$ inside every component of $\mathcal{L}_{m+n-1}$ performing a deformation of z^{m-1} to $\tilde{q} \in [q]$, which will be shown in what follows.

Let us first show that given p and $\tilde{q}$ the deformation of z^{m-1} to $\tilde{q}$ keeps the roots and critical points of $\tilde{p} = z^{-1}p(z)\tilde{q}(z)$ in the same neighbourhood as those of $\tilde{q}$.

Lemma 1. *Let $z_1, \dots, z_n \in \mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and $p(z) = \prod_{k=1}^n (z - z_k)$. Then*

$$\left|\frac{p'(z)}{p(z)}\right| = \left|\sum_{k=1}^n \frac{1}{z - z_k}\right| > \frac{n}{2}, \quad \text{on } \mathbb{T} = \partial\mathbb{D}.$$

Proof. Indeed, consider the mapping $w = \frac{1}{1-z}$ of $\mathbb{D}$ onto the right half-plane $\{w \in \mathbb{C} \colon \operatorname{Re} w > \frac{1}{2}\}$. Then assume $\zeta_k \in \mathbb{D}$ we have

$$\left|\frac{1}{1-\zeta_k}\right| \geq \operatorname{Re} \frac{1}{1-\zeta_k} > \frac{1}{2} \quad \text{and} \quad \left|\sum_{k=1}^n \frac{1}{1-\zeta_k}\right| \geq \operatorname{Re} \sum_{k=1}^n \frac{1}{1-\zeta_k} > \frac{n}{2}.$$

If $|z| = 1$, then

$$\left|\sum_{k=1}^n \frac{1}{z - z_k}\right| = \left|\sum_{k=1}^n \frac{\bar{z}}{1 - \bar{z}\, z_k}\right| = \left|\sum_{k=1}^n \frac{1}{1 - \bar{z}\, z_k}\right|.$$

Substituting $\bar{z} z_k = \zeta_k$ we finish the proof. □

Corollary 1. *Let $z_1, \dots, z_n \in \{z \in \mathbb{C} \colon |z| < \varepsilon\}$ and $p(z) = \prod_{k=1}^n (z - z_k)$. Then*

$$\left|\frac{p'(z)}{p(z)}\right| = \left|\sum_{k=1}^n \frac{1}{z - z_k}\right| > \frac{n}{2\varepsilon}, \quad \text{on } \{z \in \mathbb{C} \colon |z| = \varepsilon\}.$$

Theorem 5. *Given lemniscate generic polynomials $p_n(z) = z\prod_{k=1}^{n-1}(z - z_k)$ and $q_m(z) = z\prod_{j=1}^{m-1}(z - w_j)$ with*

$$|w_j| < \varepsilon = \frac{m}{2(n-1) + m} \min_k |z_k|,$$

the critical points of q_m and $m-1$ critical points of $P(z) = z^{-1}\, q_m p_n$ 'inherited' from q_m lie within the same disk $|z| < \varepsilon$.

Proof. Indeed,

$$\left|\frac{p_n'(z)}{p_n(z)} - \frac{1}{z}\right| = \left|\sum_{k=1}^{n-1} \frac{1}{z - z_k}\right| \leq \sum_{k=1}^{n-1} \frac{1}{|z - z_k|} \leq \frac{n-1}{\min_k \min_{|z|=\varepsilon}\{|z - z_k|\}}.$$

Using the simple identities $\min_{|z|=\varepsilon}\{|z-z_k|\} = |z_k| - \varepsilon$ and

$$\frac{n-1}{\min_k\{|z_k|-\varepsilon\}} = \frac{m}{2\varepsilon}$$

for $\varepsilon = \frac{m}{2(n-1)+m}\min_k |z_k|$ we conclude by Corollary 1 that

$$\left|\frac{p_n'(z)}{p_n(z)} - \frac{1}{z}\right| \leq \frac{m}{2\varepsilon} < \left|\frac{1}{z} + \sum_{j=1}^{m-1}(z-w_j)\right| = \left|\frac{q_m'(z)}{q_m(z)}\right|$$

on the circle $\{|z|=\varepsilon\}$. So,

$$\left|\frac{1}{z}q_m p_n' - \frac{1}{z^2}q_m p_n\right| < \left|\frac{1}{z}q_m' p_n\right|,$$

and Rouchè's theorem implies that the holomorphic functions $P'(z)$ and $\frac{1}{z}q_m' p_n$ have the same number of zeros inside the disk $\{|z|<\varepsilon\}$. Since the function $\frac{1}{z}p_n$ has no zeros in this disk then $P'(z)$ and q_m' have the same number of zeros there which proves the theorem. □

Next, we note a conic-like structure of the sets $\mathcal{L}_n$, $\mathcal{L}_m$ and $\mathcal{L}_{n+m-1}$.

Theorem 6. *Let a lemniscate generic polynomial $p_n(z) = z\prod_{k=1}^{n-1}(z-z_k)$ belong to a connected component $E' \subset \mathcal{L}_n$, and let $q_m(z) = z\prod_{j=1}^{m-1}(z-w_j)$ belong to a connected component $E'' \subset \mathcal{L}_m$. There is a small deformation of the polynomial $z^{m-1}p_n(z)$ such that the resulting polynomial $P(z)$ belongs to a connected component $E''' \subset \mathcal{L}_{n+m-1}$ such that its projection to $\mathcal{L}_n$ is from E' and its projection to $\mathcal{L}_m$ is from E''.*

Proof. Following [12] define a map $\psi\colon \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^{n-1} \to V_n$ by

$$\psi(a_n, a_0, y) = na_n\left(\int \prod_{k=1}^{n-1}(z-y_k)dz\right) + a_0, \quad a_n \in \mathbb{C}^*,\, a_0 \in \mathbb{C},\, y \in \mathbb{C}^{n-1},$$

where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, V_n is the space of polynomials of degree n, and y_k, $k = 1,\dots,n-1$ are the critical points of the polynomial p_n. The set $\psi^{-1}(V_n\setminus\mathcal{L}_n)$ is a real hypersurface whose equation is $\prod_{i<j}|P_y(y_i)| - |P_y(y_j)| = 0$, where

$$P_y(z) = n\left(\int \prod_{k=1}^{n-1}(z-y_k)dz\right).$$

All connected components of $\mathcal{L}_n$ have a unique common point at their boundaries that corresponds to the polynomial z^n. So starting from this point we can enter any of them. The polynomial $z^k p_n(z)$, $p_n \in \mathcal{L}_n$, is in the boundary of $\mathcal{L}_{n+k}$ and $\psi^{-1}(z^k p_n(z)) \in \partial\psi^{-1}(\mathcal{L}_{n+k})$, $\psi^{-1}(\mathcal{L}_{n+k}) = \psi^{-1}(\mathcal{L}_n)\times\psi^{-1}(\mathcal{L}_k)$, and there is a projection to $\psi^{-1}(\mathcal{L}_n)$ defined by $z^k p_n(z) \to p_n(z)$. Since z^k is a common boundary point for all connected components of $\mathcal{L}_k$, there exists a path connecting the boundary point $z^k \in \partial \mathcal{L}_k$ and an arbitrary point in every connected component of $\mathcal{L}_k$.

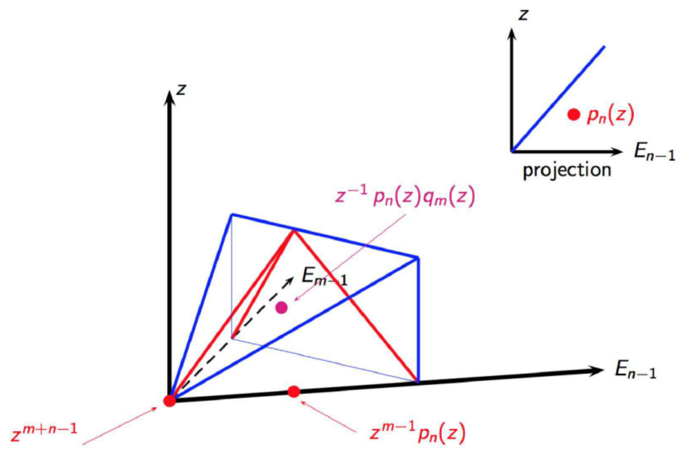

FIGURE 4. Projections of $\psi^{-1}(z^{-1}p_nq_m)$

Applying now these arguments to the polynomials $P(z)$ and $z^{m-1}p_n(z)$ yields the conclusion of the theorem. (Figure 4 gives a schematic idea how these projections are realized for $\psi^{-1}(\mathcal{L}_{n+m-1})$.) $\square$

4.4. **Operad construction.** We construct a non-unitary operad on central binary trees with labelled ends with numbers assigned to them. Let us note that central balanced trees representing lemniscate generic polynomials are in particular central and binary.

A generic element of $\mathcal{O}(k)$ is a central binary tree T with k labelled ends, together with a sequence of admissible pairs of numbers $((l_1, v_1), \dots, (l_k, v_k))$. The tuple $(v_1, \dots, v_k)$ consists of labels of the ends of T. Let s be the distance from the end v_j to the closest vertex with valence greater than one, which is either the center or one of the nodes. If $l_j \geq -s+1$, we call the number l_j and the pair (l_j, v_j) admissible.

Example 1. The tree T on figure 5 is a central balanced tree of length 2. The center of the tree (marked by white) is the closest to the end 3 vertex of valence larger than one. The distance s between the end 3 and the center is 2. This means that an admissible number assigned to the vertex 3 must be be greater or equal -1. Vertices 1 and 2 can come in pair with some non-negative integers. For example, T together with $((0,1),(0,2),(1,3))$ is an element of $\mathcal{O}(3)$.

We define the identity $1 \in \mathcal{O}(1)$ to be one labelled vertex with 0 assigned to it.

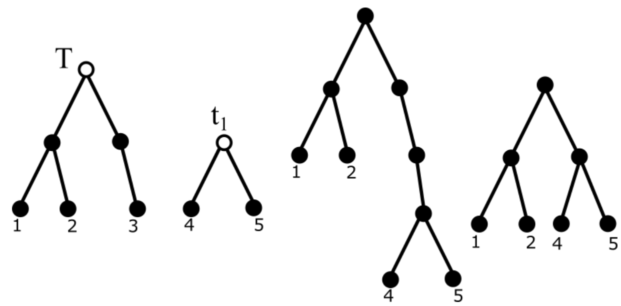

FIGURE 5. Attaching a tree to an end: example

In what follows we define the non-unitary operad composition operation. Given an element of $\mathcal{O}(k)$ which is a central binary tree T with k pairs

$$\left((l_1^0, v_1^0), \ldots, (l_k^0, v_k^0)\right)$$

and k elements $t_1 \in \mathcal{O}(j_1)$ with pairs $((l_1, v_1), \ldots, (l_{j_1}, v_{j_1}))$, $\ldots$, $t_k \in \mathcal{O}(j_k)$ with pairs $((l_{n-j_k+1}, v_{n-j_k+1}), \ldots, (l_n, v_n))$, where $n = j_1 + \cdots + j_k$. The result $\widetilde{T} = (T; t_1, \ldots, t_k)$ of the composition of T and $t_1, \ldots, t_k$ must be an element of $\mathcal{O}(n)$, i.e. a central binary tree with n ends together with n admissible pairs.

Let us first describe the construction of the tree $\widetilde{T}$. As index p varies from 1 to k, we attach the center of the tree t_p to the end v_p^0 "at distance" l_p^0 from the end. Namely, if $l_p^0 \geq 0$, we attach the center of tree t_p to a chain of l_p^0 edges, and then we attach the resulting tree to the end v_p^0. If, in turn, the number l_p^0 is negative, we erase a chain of $|l_p^0|$ edges containing v_p^0 and attach the center of the tree t_p to the vertex which connected the deleted chain and the rest of the tree.

Example 2. Let T on figure 5 have pairs $((1,3),(0,1),(0,2))$. The second from the right tree on figure 5 is the result of attaching the tree t_1 to the end 3 "at distance" 1. Suppose now T has pairs $((-1,3),(0,1),(0,2))$, the result of attaching the tree t_1 is the tree on the right hand side of the figure.

Remark 3. *Let β be the closest to v_j^0 vertex of valence larger than one in T. An admissible number l_j^0 is defined so that we never erase β and have a freedom to attach a tree closer or further from β.*

When all $t_1, \ldots, t_k$ are attached to T, the resulting tree $\overline{T}$ is not necessarily central. We transform the resulting tree into a central tree $\widetilde{T}$.

The former center v of T will be the center of the tree $\widetilde{T}$, i.e. the distance from the center v of of $\widetilde{T}$ to all the ends of $\widetilde{T}$ must be the same. The possible distances from v to the ends of $\overline{T}$ are $S_p = |T| + l_p^0 + |t_p|$, where $|T|$ and $|t_p|$ are lengths of T and t_p respectively where p varies from 1 to k. The maximum S among S_p, $1 \leq p \leq k$, will be the length of $\widetilde{T}$, i.e.

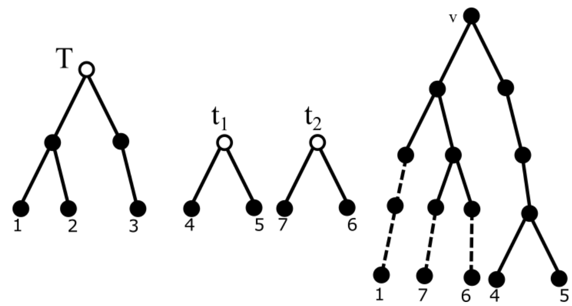

FIGURE 6. Composition of trees: example

$$|\widetilde{T}| = |\,(T; t_1, \dots, t_k)\,| = S = |T| + \max_{1\le j\le k}\left(l_j^0 + |t_j|\right). \tag{5}$$

If an end of $\overline{T}$ is at distance S_p from v, we add a chain of

$$\delta_p := S - S_p = \max_{1\le j\le k}\left(l_j^0 + |t_j|\right) - (l_p^0 + |t_p|)$$

edges to the end, the new end inherits the label.

Example 3. Let T on figure 6 have pairs $((1,3),(0,2),(0,1))$. Let us construct the composition of the trees T and $(t_1, t_2, 1)$. We will not specify the pairs assigned to the trees t_1 and t_2 for now and focus on the construction of the tree. We attach the tree t_1 to the vertex 3 "at distance" 1, the tree t_2 to the vertex 2 and the identity element 1 to the vertex 1. The tree $\overline{T}$ is shown on the figure 6 with solid lines. The maximal distance from the ends of $\overline{T}$ to the former center v is 4. We attach a chain of 2 edges to the vertex 1 and chains of 1 edge to the vertices 7 and 6, the added edges are shown with dashed lines. The new ends inherit the labels of the old ends. The resulting tree $\widetilde{T}$ is a central tree of length 4 with 5 ends.

The resulting tree $\widetilde{T}$ has $j_1 + \dots + j_k = n$ vertices

$$v_1, \dots, v_{j_1}, v_{j_1+1}, \dots, v_{j_1+j_2}, \dots, v_{n-j_k+1}, \dots, v_n.$$

Given an integer m between 1 and n, assume $j_1 + \dots + j_{p-1} + 1 \le m \le j_1 + \dots + j_p$, for some $1 \le p \le k$. To the vertex v_m, which is inherited from t_p, we assign the number $l_m - \delta_p$, where δ_p is defined above and represents the difference between the length S of the tree $\widetilde{T}$ and the distance S_p from the vertex v_m to v in $\overline{T}$. The pairs assigned to the tree $\widetilde{T}$ are as follows:

$$\big((l_1-\delta_1, v_1), \dots, (l_{j_1}-\delta_1, v_{j_1}), (l_{j_1+1}-\delta_2, v_{j_1+1}), \dots, (l_{j_1+j_2}-\delta_2, v_{j_1+j_2}), \dots,$$
$$(l_{n-j_k+1}-\delta_k, v_{n-j_k+1}), \dots, (l_n-\delta_k, v_n)\big).$$

This concludes the construction of the composition $(T; t_1, \dots, t_k) \in \mathcal{O}(n)$.

Example 4. In the previous example we have $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = 2$. Let t_1 and t_2 have sequences $((0,4),(1,5))$ and $((1,7),(2,6))$ correspondingly. The identity element is a vertex with a pair $(0,1)$.

The tree $\widetilde{T}$ is assigned the following sequence:

$$((0,4),(1,5),(0,7),(1,6),(-2,1))\,.$$

Let us verify that the number $l_m - \delta_p$ assigned to the end v_m in $\widetilde{T} = (T; t_1, \dots, t_k)$ is admissible. The integer l_m assigned to the vertex v_m in t_p is admissible, i.e.,

$$l_m \geq -s + 1, \tag{6}$$

where s is is the distance from v_m to the closest vertex of valence greater than one in t_p. Note that the former center of t_p becomes a node in $\widetilde{T}$. The end v_m of $\widetilde{T}$ is at distance $s+\delta_p$ from the closest node of $\widetilde{T}$, and this distance is shorter than the distance to the center v of $\widetilde{T}$. An admissible number assigned to v_m in $\widetilde{T}$ should be larger or equal to $-(s+\delta_p)+1$. By construction we assign to v_m a number $l_m - \delta_p$, from (6) we obtain that $l_m \geq -s+1-\delta_p$. Therefore, the new number l_m for the vertex v_m of the composition $\widetilde{T}$ is admissible and thus the operation of composition of elements of $\mathcal{O}(k)$ is well-defined.

Theorem 7. *The sequence $\{\mathcal{O}(n)\}_{n=1}^{\infty}$ of central binary trees with labelled vertices and admissible numbers assigned to them, together with operation of composition defined above, forms a non-unitary operad.*

The operation of composition is defined in accordance with definition 1 of an operad. We only need to show that the associativity axiom holds. In order to do that consider an element $T \in \mathcal{O}(k)$, with a sequence

$$\left((l_1^0, v_1^0), \dots, (l_k^0, v_k^0)\right),$$

k elements $t_J \in \mathcal{O}(j_J)$ $1 \leq J \leq k$ with pairs

$$\left((l_{j_1+\dots+j_{J-1}+1}, v_{j_1+\dots+j_{J-1}+1}), \dots, (l_{j_1+\dots+j_J}, v_{j_1+\dots+j_J})\right),$$

$\sum_1^k j_J = n$, and n elements $\tau_L \in \mathcal{O}(m_L)$, $1 \leq L \leq n$, with sequences

$$\left((\lambda_{m_1+\dots+m_{L-1}+1}, \beta_{m_1+\dots+m_{L-1}+1}), \dots, (\lambda_{m_1+\dots+m_L}, \beta_{m_1+\dots+m_L})\right).$$

We need to prove that

$$((T; t_1, \dots, t_k); \tau_1, \dots, \tau_n) = (T; (t_1; \tau_1, \dots, \tau_{j_1}), \dots, (t_k; \tau_{n-j_k+1}, \dots, \tau_n))\,. \tag{7}$$

We denote the left hand side element by T_1 and the right hand side element by T_2.

Let us first show that the trees T_1 and T_2 coincide. Then we conclude the proof by showing that the elements T_1 and T_2 have the same sets of pairs.

Lemma 2. *The trees T_1 and T_2 coincide.*

Proof. Let us show that the trees T_1 and T_2 have the same length

First, we calculate $|T_1| = |\,((T; t_1, \ldots, t_k); \tau_1, \ldots \tau_n)\,|$.

By (5) the length $|\,(T; t_1, \ldots, t_k)\,|$ is given by

$$|\,(T; t_1, \ldots, t_k)\,| = |T| + \max_{1\le j\le k}\{l^0_j + |t_j|\}.$$

The tree $(T; t_1, \ldots, t_k)$ has vertices

$$v_1, \ldots, v_{j_1}, \ldots, v_{n-j_k+1}, \ldots, v_n$$

with numbers

$$l_1 - (\max_{1\le j\le k}\{l^0_j + |t_j|\} - (l^0_1 + |t_1|)), \ldots,$$
$$l_{j_1} - (\max_{1\le j\le k}\{l^0_j + |t_j|\} - (l^0_1 + |t_1|)), \ldots,$$
$$l_{n-j_k+1} - (\max_{1\le j\le k}\{l^0_j + |t_j|\} - (l^0_k + |t_k|)), \ldots,$$
$$l_n - (\max_{1\le j\le k}\{l^0_j + |t_j|\} - (l^0_k + |t_k|)).$$

Thus the length of $((T; t_1, \ldots, t_k); \tau_1, \ldots \tau_n)$ is

$$|T| + \max_{1\le j\le k}\{l^0_j + |t_j|\} + \max_{1\le m\le n}\{l_m + |\tau_m|\} =$$

$$= |T| + \max_{1\le m\le n}\{|\tau_m| + l_m + l^0_{j(m)} + |t_{j(m)}|\}. \tag{8}$$

Here $j(m) = 1$ if $m = 1, \ldots, j_1$, $j(m) = 2$ if $m = j_1 + 1, \ldots, j_1 + j_2$, and so on. Given m, the tree τ_m is glued to an end of some tree t_j, so $j = j(m)$ is the index of this tree, which is uniquely determined by m.

Let us calculate now $|T_2| = |\,(T; (t_1; \tau_1, \ldots \tau_{j_1}), \ldots, (t_k; \tau_{n-j_k+1}, \ldots, \tau_n))\,|$.

$$|T_2| = |T| + \max_{1\le p\le k}\{l^0_p + |t_p| + \max_{j_1+\cdots+j_{p-1}+1\le m\le j_1+\cdots+j_p}\{l_m + |\tau_m|\}\}. \tag{9}$$

Given $1 \le p \le k$, we define $m(p)$, $1 \le m \le n$, to be indices of τ_m that are glued to the tree t_p. Given p, the values of $m(p)$ are integers between $j_1 + \cdots + j_{p-1} + 1$ and $j_1 + \cdots + j_p$. The expression (9) can be rewritten as

$$|T_2| = |T| + \max_{1\le p\le k}\max_{m(p)}\{l^0_p + |t_p| + l_m + |\tau_m|\}. \tag{10}$$

The expressions (10) and (8) coincide. Therefore, the trees T_1 and T_2 have the same length.

Let us show now that the trees T_1 and T_2 coincide.

Let us fix $1 \le p \le k$ and let v be the center of t_p. The center of t_p is glued to the end v^0_p of T "at distance" l^0_p during the construction of both T_1 and T_2.

We denote by β a vertex in T of valence greater than one closest to v^0_p. The vertex β is either the center, or a node of T. The end v^0_p is attached to β by a chain of edges. During construction of T_1 and T_2 we replace this chain of edges with a new one and attach the center v of t_p to it. It is clear that the distance from v to the center of T in both cases is the same and equal to $|T| + l_p^0$. We can conclude that the trees t_p, $1 \leq p \leq k$ are attached to the same positions in T_1 and T_2.

Let us now denote by b the center of τ_L, $1 \leq L \leq n$. Suppose τ_L is attached to the end v_L of t_J for some J between 0 and k. We denote by V the vertex of t_J of valence larger than one, that is closest to v_L. The end v_L is attached to V by a chain of edges. We replace this chain of edges with another one and attach the center b of τ_L to the new chain. The distance from b to V is $|t_J|+l_L$ both, in $(T; (t_1; \tau_1, \dots \tau_{j_1}), \dots, (t_k; \tau_{n-j_k+1}, \dots, \tau_n))$, and $((T; t_1, \dots, t_k); \tau_1, \dots \tau_n)$. We can conclude that the trees τ_L, $1 \leq L \leq n$ are attached to the same positions in T_1 and T_2.

In addition, the lengths of T_1 and T_2 coincide, thus the trees T_1 and T_2 coincide.

□

The following lemma concludes the proof of the theorem.

Lemma 3. *The pairs assigned to T_1 and T_2 are identical.*

Proof. First we calculate the sequences for the left hand side of (7).

The composition $(T; t_1, \dots, t_k)$ is an element in $\mathcal{O}(n)$ with the sequence

$$\begin{gathered}((l_1 - (S - S_1), v_1), \dots, (l_{j_1} - (S - S_1), v_{j_1}),\\ (l_{j_1+1} - (S - S_2), v_{j_1+1}), \dots, (l_{j_1+j_2} - (S - S_2), v_{j_1+j_2}), \dots,\\ (l_{n-j_k+1} - (S - S_k), v_{n-j_k+1}), \dots, (l_n - (S - S_k), v_n)),\end{gathered}$$

where $S_J = |T| + l_J^0 + |t_J|$, $1 \leq J \leq k$; S is the maximum of S_J. S is the length of the tree $(T; t_1, \dots, t_k)$.

Composition $((T; t_1, \dots, t_k); \tau_1, \dots \tau_n)$ is an element in $\mathcal{O}(m_1 + \cdots + m_n)$. Let L be an integer between 1 and n. Recall that $n = j_1 + \cdots + j_k$ and suppose $j_1 + \cdots + j_{p-1} + 1 \leq L \leq j_1 + \cdots + j_p$ for some $1 \leq p \leq k$. Note that $1 \leq L - (j_1 + \cdots + j_{p-1}) \leq j_p$. We attach the tree τ_L to the end v_L "at distance" $l_L - (S - S_p)$. We define

$$\Delta_L = |(T; t_1, \dots, t_k)| + l_L - (S - S_p) + |\tau_L|,$$

which can be rewritten as

$$\Delta_L = l_L + S_p + |\tau_L|,$$

and let Δ denote the maximum of Δ_L. We obtain

$$\Delta = |\, ((T; t_1, \dots, t_k); \tau_1, \dots \tau_n) \,|.$$

The composition $((T; t_1, \dots, t_k); \tau_1, \dots \tau_n)$ has the following sequence:

$$((\lambda_1-(\Delta-\Delta_1),\beta_1),\dots,(\lambda_{m_1}-(\Delta-\Delta_1),\beta_{m_1}),$$
$$(\lambda_{m_1+1}-(\Delta-\Delta_2),\beta_{m_1+1}),\dots,(\lambda_{m_1+m_2}-(\Delta-\Delta_2),\beta^2_{m_1+m_2}),\dots,$$
$$(\lambda_{m_1+\cdots+m_{n-1}+1}-(\Delta-\Delta_n),\beta_{m_1+\cdots+m_{n-1}+1}),\dots,$$
$$(\lambda_{m_1+\cdots+m_n}-(\Delta-\Delta_n),\beta_{m_1+\cdots+m_n}))\,.$$

Let R be an integer between 1 and $m_1+\cdots+m_n$. Suppose

$$m_1+\cdots+m_{L-1}+1\le R\le m_1+\cdots+m_L$$

for some $1\le L\le n$ and, also, assume

$$j_1+\cdots+j_{p-1}+1\le L\le j_1+\cdots+j_p$$

for some $1\le p\le k$. The $R-$th pair in the sequence assigned to T_1 has form $(\lambda_R-(\Delta-\Delta_L),\beta_R)$, where

$$\Delta-\Delta_L=|\,((T;t_1,\dots,t_k);\tau_1,\dots\tau_n)\,|-(l_L+S_p+|\tau_L|)=$$
$$=|\,((T;t_1,\dots,t_k);\tau_1,\dots\tau_n)\,|-\left(l_L+|T|+l^0_p+|t_p|+|\tau_L|\right).$$

Let us now write down the sequence of pairs for T_2.

Let $1\le L\le n$ and also assume that $j_1+\cdots+j_{p-1}+1\le L\le j_1+\cdots+j_p$ for some $1\le p\le k$. We define $\sigma_L=|t_p|+l_L+|\tau_L|$, Σ_p is the maximum of σ_L, where $j_1+\cdots+j_{p-1}+1\le L\le j_1+\cdots+j_p$, it is the length of the tree $(t_p;\tau_{j_1+\cdots+j_{p-1}+1},\dots,\tau_{j_1+\cdots+j_p})$.

Given p between 1 and k, the element $(t_p;\tau_{j_1+\cdots+j_{p-1}+1},\dots,\tau_{j_1+\cdots+j_p})$ has the sequence

$$\Big(\Big(\lambda_{m_1+\cdots+m_{j_1+\cdots+j_{p-1}}+1}-(\Sigma_p-\sigma_{j_1+\cdots+j_{p-1}+1}),\beta_{m_1+\cdots+m_{j_1+\cdots+j_{p-1}}+1}\Big),\dots,$$

$$\Big(\lambda_{m_1+\cdots+m_{j_1+\cdots+j_{p-1}+1}}-(\Sigma_p-\sigma_{j_1+\cdots+j_{p-1}+1}),\beta_{m_1+\cdots+m_{j_1+\cdots+j_{p-1}+1}}\Big),\dots,$$

$$\Big(\lambda_{m_1+\cdots+m_{j_1+\cdots+j_p-1}+1}-(\Sigma_p-\sigma_{j_1+\cdots+j_p}),\beta_{m_1+\cdots+m_{j_1+\cdots+j_p-1}+1}\Big),\dots,$$

$$\Big(\lambda_{m_1+\cdots+m_{j_1+\cdots+j_p}}-(\Sigma_p-\sigma_{j_1+\cdots+j_p}),\beta_{m_1+\cdots+m_{j_1+\cdots+j_p}}\Big)\Big)\,.$$

We attach the center of the tree $(t_p;\tau_{j_1+\cdots+j_{p-1}+1},\dots,\tau_{j_1+\cdots+j_p})$ to the vertex v^0_p "at distance" l^0_p, as p varies between 1 and k.

We define $Q_p=|T|+l^0_p+|(t_p;\tau_{j_1+\cdots+j_{p-1}+1},\dots,\tau_{j_1+\cdots+j_p})|=|T|+l^0_p+\Sigma_p$. We define Q to be the maximum among Q_p, $1\le p\le k$; Q is the length of the tree T_2.

Let us choose R, where $1\le R\le m_1+\cdots+m_n$, and write down the $R-$th pair of T_2. Suppose $m_1+\cdots+m_{L-1}+1\le R\le m_1+\cdots+m_L$ for some

$1 \le L \le n$ and suppose $j_1 + \cdots + j_{p-1} + 1 \le L \le j_1 + \cdots + j_p$ for some $1 \le p \le k$. The $R-$th pair in the sequence for T_2 has the form

$$(\lambda_R - (\Sigma_p - \sigma_L) - (Q - Q_p), \beta_R)\,.$$

Let us rewrite

$$(\Sigma_p - \sigma_L) + (Q - Q_p) = \Sigma_p - (|t_p| + l_L + |\tau_L|) + |T_2| - (|T| + l_p^0 + \Sigma_p) =$$

$$= |T_2| - \left(|t_p| + l_L + |\tau_L| + |T| + l_p^0\right).$$

We can see that the $R-$th pair for $((T; t_1, \ldots, t_k); \tau_1, \ldots \tau_n)$ coincides with the $R-$th pair for $(T; (t_1; \tau_1, \ldots \tau_{j_1}), \ldots, (t_k; \tau_{n-j_k+1}, \ldots, \tau_n))$.

□

Remark 4. *In section 4.3 we discussed the composition $[p] \circ_k [q]$ of conjugacy classes of lemniscate generic polynomials $p \in \mathcal{L}_n$ and $q \in \mathcal{L}_m$.*

Let T and t be the central balanced trees of length $n-1$ and $m-1$ which correspond to p and q respectively. Let us label the zeros of p, and respectively ends of T, by numbers from 1 to n.

Geometrically, composition $[p] \circ_k [q]$ can be described as follows. We take a neighbourhood containing the big lemniscates of q, shrink it and replace with it a small neighbourhood of the zero k of p. In terms of trees this is analogous to gluing the tree t to the vertex k of T "at distance" 0. To obtain the lemniscate configuration of $[p] \circ_k [q]$ we add extra circumferences around the zeros of p, or, in terms of trees, extend the tree (as described in section 4.4) so that it becomes a central balanced tree of length $n+m-1$.

The composition of p and q can be realized as composition of an element $T \in \mathcal{O}(n)$ with the tuple $(1, \ldots, 1, t, 1, \ldots, 1)$ of length n, there $t \in \mathcal{O}(m)$ in at the $k-$th place in the tuple, the trees T and t represent p and q, 1 is the identity in $\mathcal{O}(1)$. The polynomial fireworks can be realized through the non-unitary operad constructed in section 4.4. Thus, the goal we set in the Introduction has been achieved.

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